JOURNAL OF ALGEBRA 32, 215–233 (1974)

The Free Inverse Semigroup on Two Commuting Generators

D. B. McALISTER AND R. McFADDEN*

Department of Mathematics, Northern Illinois University, DeKalb, Illinois 60115

Communicated by G. B. Preston

Received January 14, 1973

In this paper we utilize the zig-zag representations to characterize the free inverse semigroup on two commuting generators.

1.

If S is a semigroup, there exist an inverse semigroup $I(S)$ and a homomorphism $\eta : S \to I(S)$ with the following property: given any homomorphism θ of S into an inverse semigroup T, there is a unique homomorphism $\psi : I(S) \to T$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\theta"'] & I(S) \arrow[dl, "\psi"] \\ T & \end{tikzcd}$$

commutes. The semigroup $I(S)$ is called *the free inverse semigroup on* S [2, 8].

It follows directly from the functorial properties of S^1 and $I(S)$ that $I(S^1)$ and $I(S)^1$ are naturally isomorphic. Therefore in studying relationships between S and $I(S)$, we may assume without loss of generality that S has an identity element. We shall do this throughout the paper.

Our main result is Theorem 2.16 which provides a characterization of $I(S)$ when S is the free monoid on two commuting generators. In this case, of course, $S \approx Z^+ \times Z^+$, and may be considered to be a submonoid of $Z \times Z$. We utilize the zig-zag representations introduced in [4], and the reader is referred to that paper for details.

Let G be a group and let S be a submonoid of G. Then the relation $\leqslant'$ defined on G by:

$$g \leqslant' h \text{ if and only if } h = gx \text{ for some } x \in S$$

* This research was supported by N.S.F. Grant No. GP27917.

is a quasiorder. It is compatible with multiplication on the left, and $S = \{x \in G : 1 \leqslant' x\}$. Left compatible quasiorders coincide one to one with submonoids of G.

For the remainder of this paper, when we are dealing with a submonoid S of a group G, we shall denote by $\leqslant'$ the corresponding left compatible quasiorder, and for a given left compatible quasiorder on G, write $S = \{x \in G : 1 \leqslant' x\}$. When we are dealing with an inverse semigroup S, we shall use $\leqslant$ to denote the natural partial order on $S[1]$.

Suppose now that G is a group with a left compatible quasiorder and let $\mathscr{X}$ be a set of subsets of G with the following properties:

(i) if $A \in \mathscr{X}$ and $1 \leqslant' y \leqslant' x$, where $1, x \in A$ then $y \in A$.

(ii) if $A \in \mathscr{X}$ and $x \in S \cap A$ then $x^{-1}A \in \mathscr{X}$.

Such a set $\mathscr{X}$ is called a *representing family* of subsets of G. For each $s \in S$ define a partial transformation ρ_s of $\mathscr{X}$ by

$$A\rho_s = s^{-1}A \text{ for each } A \in \mathscr{X} \text{ with } s \in A.$$

PROPOSITION 1.1 [4]. *The mapping $\rho : s \to \rho_s$ is a representation of S by one-to-one partial transformations of $\mathscr{X}$.*

The representation ρ in Proposition 1.1 is called the *zig-zag representation* of S determined by $\mathscr{X}$.

DEFINITION [4]. A subset X of G is called *solid* if $a \leqslant' x \leqslant' b$, $a, b \in X$ implies $x \in X$. A subset B of G is *solid up* from a subset A of G if $A \subseteq B$, and if $a \leqslant' y \leqslant' b$, $a \in A$, $b \in B$, imply $y \in B$.

If S is any semigroup we shall denote by S^{-1} the semigroup whose elements are x^{-1}, $x \in S$, with the multiplication $x^{-1}y^{-1} = (yx)^{-1}$. The free product [1] of S and S^{-1} will be denoted by $S * S^{-1}$; it has an involution defined by $(s_1t_1^{-1} \cdots s_rt_r^{-1})^{-1} = t_rs_r^{-1} \cdots t_1s_1^{-1}$, where, in order that $s_1t_1^{-1} \cdots t_r^{-1}$ denotes a generic element of $S * S^{-1}$, we allow either or both of s_1, t_r^{-1} to be empty.

Any homomorphism θ of S into an inverse semigroup T can be extended to a homomorphism, which we also denote by θ, of $S * S^{-1}$ into T, by defining

$$w\theta = (s_1\theta)(t_1\theta)^{-1} \cdots (s_r\theta)(t_r\theta)^{-1} \qquad \text{for} \quad w = s_1t_1^{-1} \cdots s_rt_r^{-1} \in S * S^{-1}.$$

This is the unique extension of θ to a homomorpism that preserves the involution. In particular, there is a canonical homomorphism ϕ of $S * S^{-1}$ onto $I(S)$; for $x \in S$, the image of x^{-1} under ϕ is the inverse of x in $I(S)$. Normally we shall denote $w\phi$, for $w \in S * S^{-1}$, simply by w in $I(S)$.

Since any representation ρ of S is a homomorphism of S into an inverse semigroup it can be extended to a representation, also denoted by ρ of $S * S^{-1}$.

DEFINITION. Let S be a submonoid of a group G; then $u \in S * S^{-1}$ is a *left factor* of $w \in S * S^{-1}$ if $w = uv$ for some $v \in S * S^{-1}$. The set of *reduced left factors* of w is the set

$$\{u \in G: u \in S * S^{-1} \text{ is a left factor of } w\};$$

it is denoted by rlf. w.

LEMMA 1.2 [4]. *Let $\mathscr{X}$ be a representing family of subsets of G and let ρ be the corresponding zig-zag representation; let $w = s_1t_1^{-1} \cdots s_rt_r^{-1} \in S * S^{-1}$. If $A \in \mathscr{X}$ belongs to domain ρ_w, then A is solid up from rlf. w; note $1 \in$ rlf. w.*

Conversely, if $\mathscr{X}$ is the set of all solid subsets of G which contain 1, *and A is solid up from $\{1, s_1, s_1t_1^{-1},..., s_1t_1^{-1} \cdots s_rt_r^{-1}\}$ then A is in domain ρ_w.*

COROLLARY 1.3 [4]. *Let $w = s_1t_1^{-1} \cdots s_rt_r^{-1} \in S * S^{-1}$. Then rlf. w and $\{1, s_1, s_1t_1^{-1},..., s_1t_1^{-1} \cdots s_rt_r^{-1}\}$ have the same solid hull.*

DEFINITION [4]. A *zig-zag* in G is a finite ordered sequence $\{a_0 = 1, a_1, ..., a_n\}$ of elements of G such that either $a_0 \leqslant' a_1 \geqslant' a_2 \leqslant' a_3 \cdots$ or $a_0 \geqslant' a_1 \leqslant' a_2 \geqslant' a_3 \cdots$, with the ordering part of the definition.

There is a one-to-one correspondence between $S * S^{-1}$ and the set of zig-zags in G: corresponding to $w = s_1t_1^{-1}s_2t_2^{-1} \cdots$ we have $1 \leqslant' s_1 \geqslant' s_1t_1^{-1} \leqslant' s_1t_1^{-1}s_2 \cdots$ and to $w = t_1^{-1}s_2t_2^{-1} \cdots$ we have $1 \geqslant' t_1^{-1} \leqslant' t_1^{-1}s_2 \geqslant' t_1^{-1}s_2t_2^{-1} \cdots$; conversely to $a_0 \leqslant' a_1 \geqslant' a_2 \leqslant' \cdots$ corresponds the unique word $s_1t_1^{-1}s_2 \cdots$ in $S * S^{-1}$, where $s_0 = s_1$, $a_2^{-1}a_1 = t_1$,..., and to $a_0 \geqslant' a_1 \leqslant' a_2 \cdots$ the unique word $t_1^{-1}s_2t_2^{-1} \cdots$, where $a_1^{-1} = t_1$, $s_2 = a_1^{-1}a_2$,... .

Suppose that $\mathscr{X}$ is a family of solid subsets of G which is invariant under left translation and that each zig-zag A has a closure $\bar{A}$ in $\mathscr{X}$. Then the closures of zig-zags form a *subsemilattice* of $\mathscr{X}$ with smallest member $\{\bar{1}\}$; if u, $v \in S * S^{-1}$ then $\overline{\text{rlf. } u\, u^{-1}v}$ is the least upper bound of the zig-zags corresponding to u and v. If A and B are closures of zig-zags, we shall denote by $A \vee B$ the smallest member of $\mathscr{X}$ containing A and B.

LEMMA 1.4 [4]. *Let $u, v \in S * S^{-1}$; then*

$$\text{solid hull rlf. } u\, v = (\text{solid hull rlf. } u) \vee u(\text{solid hull rlf. } v)$$

where $\vee$ denotes the join of solid sets.

COROLLARY 1.5 [4]. *Let $\mathscr{X}$ be a family of subsets of G which is invariant* *under left translation by G. Suppose further that, for each zig-zag A of G, there is a smallest member $\bar{A}$ of $\mathscr{X}$ containing A. If $u, v \in S * S^{-1}$, then*

$$\overline{\text{rlf. } uv} = \overline{\text{rlf. } u} \vee u\, \overline{\text{rlf. } v},$$

where if A and B are zig-zags, $\bar{A} \vee \bar{B} = \overline{A \cup B}$.

2. P-Semigroups and $I(Z^+ \times Z^+)$

We now describe the construction for a family of inverse semigroups which we call P-semigroups. These semigroups are in easily manageable form, and a wide class of inverse semigroups may be described in terms of P-semigroups [3, 4]. Again, details may be found in [4].

Let $\mathscr{X}$ be a partially ordered set and let $E \in \mathscr{X}$ be such that $\{A \in \mathscr{X}: E \leqslant A\}$ is a $\vee$-semilattice. Let G be a group which acts on $\mathscr{X}$, on the left, by order automorphisms and let $P = P(\mathscr{X}, G, E)$ be the set of all pairs (A, g) where $g \in G$, $A \in \mathscr{X}$ and $E \leqslant A$, $g^{-1}A$. Define

$$(A, g)(B, h) = (A \vee gB, gh)$$

for $(A, g), (B, h) \in P$.

Proposition 2.1 [4]. *$P = P(G, \mathscr{X}, E)$ is an inverse semigroup. The idempotents are the elements $(A, 1)$ with $E \leqslant A \in \mathscr{X}$; they form a semilattice antiisomorphic to $\{A \in \mathscr{X}: E \leqslant A\}$. For each $(A, g) \in P$, $(A, g)^{-1} = (g^{-1}A, g^{-1})$.*

Now let $\mathscr{X}$ be the family of all solid subsets of a left quasiordered group G. Clearly $\mathscr{X}$ is invariant under left translation by G, and for each finite subset A of G, there is a smallest member $\bar{A}$ of $\mathscr{X}$ which contains A. We therefore have the following special case of Theorem 2.3 of [4].

Theorem 2.2. *Let $\mathscr{X}$ be the family of all solid subsets of a left quasiordered group G. Then the mapping $\theta : S * S^{-1} \to P(G, \mathscr{X}, \{\bar{1}\})$ defined by $w\theta = (\overline{\text{rlf. } w}, w)$ is a homomorphism.*

For the particular case when G is the free group on two commuting generators, we shall in fact prove that the inverse hull of $S\theta$ in P is the free inverse semigroup on two commuting generators. To do this, we first describe the inverse hull of $S\theta$. We denote by $\mathscr{Z}$ the family of all left translates of closures of zig-zags in G. One may show [4] that $P(G, \mathscr{Z}, \{\bar{1}\})$ is well defined, and we then have the following particular case of Theorem 2.5 of [4].

Theorem 2.3. *Let G be the free group on two commuting generators with* *the usual partial ordering and let $\mathscr{X}$ be the family of all left translates of closures of zig-zags in G. Then the mapping $\theta : S * S^{-1} \to P(G, \mathscr{X}, \{\overline{1}\})$ defined by $w\theta = (\overline{\text{rlf.}\, w}, w)$ is a homomorphism of $S * S^{-1}$ onto P.*

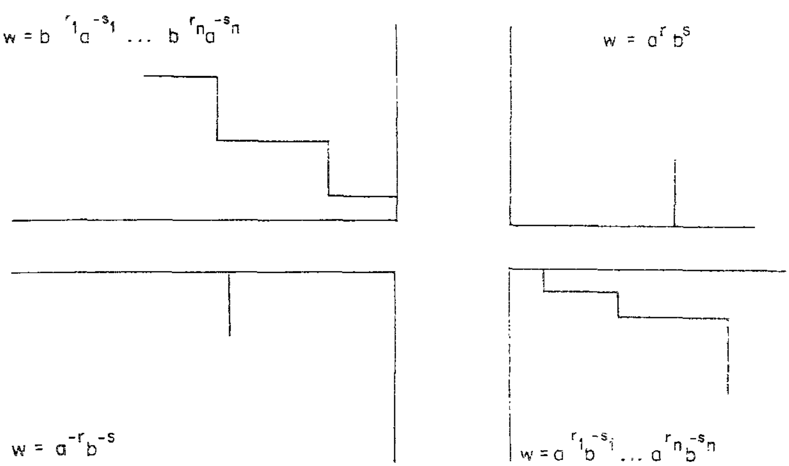
For the remainder of this paper, S will denote the free monoid on two commuting generators a and b. Every element of S is of the form $a^r b^s$ for some pair of nonnegative integers r and s, and $S \approx Z^+ \times Z^+$. Since a and b commute, so do their inverses a^{-1} and b^{-1} in $I(S)$. We shall denote by G the free group on two commuting generators, and clearly we may assume that S is a submonoid of $G \approx Z \times Z$. By $\leqslant'$ we shall mean the usual partial ordering on G, and then $S = \{x \in G : 1 \leqslant' x\}$. Note that G is the maximum group homomorphic image of $I(S)$ [1].

If w is a word in $S * S^{-1}$ in which no two adjacent syllables [1] contain the same symbol, then w must be one of the following four forms:

$$w = a^r b^s, \qquad w = a^{r_1} b^{-s_1} \cdots a^{r_n} b^{-s_n},$$

$$w = a^{-r} b^{-s}, \qquad w = b^{r_1} a^{-s_1} \cdots b^{r_n} a^{-s_n},$$

where r, s, r_i, s_i, $1 \leqslant i \leqslant n$ are nonnegative integers. We shall call words of this form, and also their canonical images in $I(S)$, *reduced words*. Since $S \approx Z^+ \times Z^+$, we may illustrate words in $S * S^{-1}$ by diagrams in the plane. For example, the four types of reduced words, and their corresponding zig-zags, may be illustrated by



LEMMA 2.4. *Let $w \in S * S^{-1}$. Then in $I(S)$, w may be written in the form ev where $e^2 = e$ and v is reduced.*

Proof. The proof is by induction on the length n of w in $S * S^{-1}$. Clearly the result holds if $n = 1$. Suppose it holds for words of length less than $n > 1$, that w has length n, and that two adjacent syllables of w have a symbol x in common, where $x \in \{a, b, a^{-1}, b^{-1}\}$, so that w may be written as $w = uxx^{-1}v$ in $S * S^{-1}$, where length $uv < n$. Then in $I(S)$ we have $w = (ux)(ux)^{-1}uv$, and by induction, $uv = ev'$ where $e^2 = e$ and v' is reduced. Therefore $w = (ux)(ux)^{-1}ev'$, v' reduced, as required.

Let T be an arbitrary inverse semigroup, and let σ denote the minimum group congruence on T. If the idempotents of T form a σ-class, we say that T is a *proper* inverse semigroup. Proper inverse semigroups are exactly those inverse semigroups in which $aa^{-1} = bb^{-1}$ and $(a, b) \in \sigma$ imply $a = b$ [4].

PROPOSITION 2.5. *$I(S)$ is a proper inverse semigroup.*

Proof. Suppose $(w, 1) \in \sigma$ in $I(S)$. By Lemma 2.4, we may write $w = ev$ where $e^2 = e$ and v is reduced. Clearly v reduces to 1 in G. If $v = a^r b^s$ then $r = s = 0$; if $v = a^{r_1}b^{-s_1} \cdots a^{r_n}b^{-s_n}$ then $a^{r_1+\cdots+r_n} = b^{s_1+\cdots+s_n}$ in G, so $r_i = s_i = 0$, $1 \leqslant i \leqslant n$. In each case $v = 1$ and w is idempotent. The other two cases are exactly analogous, whence the result.

LEMMA 2.6. *Every idempotent of $I(S)$ is of the form*

$$ww^{-1} = \bigwedge w_j w_j^{-1}$$

for some positive integer m and reduced words w_j, $1 \leqslant j \leqslant m$.

Proof. For $e^2 = e \in I(S)$ define $l(e)$ to be the length n of the shortest word $w \in S * S^{-1}$ for which $e = ww^{-1}$. The proof is by induction on $l(e)$. If $l(e) = 1$ the result is obvious.

Suppose the result holds for idempotents f in $I(S)$ for which $l(w) \leqslant n - 1$, and let $e = ww^{-1}$ where $l(e) =$ length $w = n > 1$. Either w is reduced, and we have finished, or two adjacent syllables of w have a symbol in common. In this case we may write $w = uxx^{-1}v = (ux)(ux)^{-1}uv$, where $x \in \{a, b, a^{-1}, b^{-1}\}$ and length $ux < n$, length $uv < n$. The result now follows by a double application of our induction hypothesis.

The fact that $uxx^{-1}v = (ux)(ux)^{-1}uv$ for u, v, x in any inverse semigroup T was useful in proving Lemmas 2.5 and 2.6. Further,

$$(uxx^{-1}v)(uxx^{-1}v)^{-1} = (ux)(ux)^{-1} \wedge (uv)(uv)^{-1}, \tag{1}$$

and this will be used extensively in what follows. The calculations involved can be simplified considerably by using the following notation.

DEFINITION. Let T be an inverse semigroup. For $u, v \in T$ define

$$u * v = (v^{-1}u)(v^{-1}u)^{-1}. \tag{2}$$

Then $*$ determines an action of T by order endomorphisms on the semilattice E_T of idempotents of T, for if $e \in E_T$ then $e * v = v\, e\, v^{-1}$. [6] One verifies easily that $e * uv = (e * u) * v$, and that $uu^{-1} * v = u * v$.

One of our most common problems will be to simplify $u * v$ for reduced words u and v in $I(S)$, and for this we shall use (1) frequently. In doing so we need only keep track of the left half w of an idempotent ww^{-1}, and this can be done more easily by writing (1) as $uxx^{-1}v \equiv ux \wedge uv$, and (2) as $u * v \equiv v^{-1}u$.

LEMMA 2.7. *Let $z = a^{r_1}b^{-s_1} \cdots a^{r_n}b^{-s_n}$, $n \geqslant 1$, be reduced in $I(S)$, and let k be a nonnegative integer. Let m be the greatest integer for which $\sum_{i=1}^{m} r_i \leqslant k$, where we assume $\sum_{i=1}^{0} r_i = 0$. Then*

$$z * a^k \equiv a^{-k} \wedge a^{-(k-r_1)}b^{-s_1} \wedge \cdots \wedge a^{-(k-\sum^m r_i)}b^{-\sum^m s_i}$$
$$\wedge\, b^{-\sum^m s_i}a^{\sum^{m+1} r_i - k}b^{-s_{m+1}}a^{r_{m+2}} \cdots a^{r_n}b^{-s_n},$$

where if $m = n$ the last term on the right is not present.

Proof. The proof is by induction on m. If $m = 0$ then $k \leqslant r_1$ and

$$z * a^k \equiv a^{-k}a^k a^{r_1-k}b^{-s_1} \cdots a^{r_n}b^{-s_n}$$
$$\equiv a^{-k} \wedge a^{r_1-k}b^{-s_1} \cdots a^{r_n}b^{-s_n},$$

noting that for $k = r_1$ we have $b^{-s_1} \wedge b^{-s_1}a^{r_2} \cdots a^{r_n}b^{-s_n} = b^{-s_1}a^{r_2} \cdots a^{r_n}b^{-s_n}$.

Suppose the result holds for m or fewer syllables and let $m + 1 \leqslant n$ be the greatest integer for which $\sum^{m+1} r_i \leqslant k$. Write $k - \sum^m r_i = r$, and note that $r \geqslant r_{m+1}$, while $z * a^k = z * a^{k-r} * a^r$. Since $k - r = \sum^m r_i$, the inductive hypothesis implies

$$z * a^{k-r} \equiv a^{-(k-r)} \wedge a^{-(k-r-r_1)}b^{-s_1} \wedge \cdots \wedge a^{-(k-r-\sum^m r_i)}b^{-\sum^m s_i}$$
$$\wedge\, b^{-\sum^m s_i}a^{r_{m+1}} \cdots a^{r_n}b^{-s_n}.$$

If $m \leqslant n - 2$ we claim that

$$(b^{-\sum^m s_i}a^{r_{m+1}}b^{-s_{m+1}} \cdots a^{r_n}b^{-s_n}) * a^r$$
$$\equiv a^{-(k-\sum^m r_i)}b^{-\sum^m s_i} \wedge a^{-(k-\sum^{m+1} r_i)}b^{-\sum^{m+1} s_i} \wedge b^{-\sum^{m+1} s_i}a^{\sum^{m+2} r_i - k} \cdots a^{r_n}b^{-s_n}. \quad (3)$$

From (3) it follows that

$$z * a^k = z * a^{k-r} * a^r$$
$$\equiv a^{-k} \wedge a^{-(k-r_1)}b^{-s_1} \wedge \cdots \wedge a^{-(k-\sum^{m+1} r_i)}b^{-\sum^{m+1} s_i}$$
$$\wedge\, b^{-\sum^{m+1} s_i}a^{\sum^{m+2} r_i - k} \cdots a^{r_n}b^{-s_n},$$

and the result holds in this case. The argument for $m = n - 1$ is easily deduced from the above, and to complete the proof of the lemma we need only prove (3). Since $r \geqslant r_{m+1}$,

$$\begin{aligned}
(b^{-\Sigma^m s_i} a^{r_{m+1}} b^{-s_{m+1}} \cdots a^{r_n} b^{-s_n}) * a^r
&\equiv b^{-\Sigma^m s_i} a^{-r+r_{m+1}} a^{-r_{m+1}} a^{r_{m+1}} b^{-s_{m+1}} \cdots a^{r_n} b^{-s_n} \\
&\equiv b^{-\Sigma^m s_i} a^{-r} \wedge b^{-\Sigma^m s_i} a^{-r+r_{m+1}} b^{-s_{m+1}} \cdots a^{r_n} b^{-s_n} \\
&\equiv b^{-\Sigma^m s_i} a^{-r} \wedge b^{-\Sigma^{m+1} s_i} a^{-r+r_{m+1}} \wedge b^{-\Sigma^{m+1} s_i} a^{-r+r_{m+1}+r_{m+2}} \cdots a^{r_n} b^{-s_n} \\
&= a^{-(k-\Sigma^m r_i)} b^{-\Sigma^m s_i} \wedge a^{-(k-\Sigma^{m+1} r_i)} b^{-\Sigma^{m+1} s_i} \wedge b^{-\Sigma^{m+1} s_i} a^{\Sigma^{m+2} r_i - k} \cdots a^{r_n} b^{-s_n}
\end{aligned}$$

since $-r + r_{m+1} + r_{m+2} = -k + \sum^{m+2} r_i$.

LEMMA 2.8. *Let $z = a^{r_1} b^{-s_1} \cdots a^{r_n} b^{-s_n}$, $n \geqslant 1$, be reduced in $I(S)$, and let k be a nonnegative integer. Let m be the greatest integer for which $\sum_{i=1}^{m} s_i \leqslant k$, where we assume $\sum_{i=1}^{0} s_i = 0$. Then*

if $m = n$,

$$z * b^{-k} \equiv a^{r_1} b^k \wedge a^{r_1+r_2} b^{k-s_1} \wedge \cdots \wedge a^{\Sigma^n r_i} b^{k-\Sigma^n s_i},$$

and if $m \leqslant n - 1$,

$$\begin{aligned}
z * b^{-k} \equiv\ & a^{r_1} b^k \wedge a^{r_1+r_2} b^{k-s_1} \wedge \cdots \wedge a^{\Sigma^{m+1} r_i} b^{k-\Sigma^m s_i} \\
& \wedge a^{\Sigma^{m+1} r_i} b^{-(\Sigma^{m+1} s_i - k)} a^{r_{m+2}} \cdots a^{r_n} b^{-s_n},
\end{aligned}$$

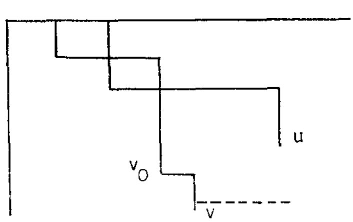
where if $m = n - 1$ the last term on the right is not present.

Apart from the fact that b^k commutes with a^{r_1}, the proof is exactly analogous to that of Lemma 2.7, and we omit it.

For $u = a^{r_1} b^{-s_1} \cdots a^{r_n} b^{-s_n} \in I(S)$, define $a(u) = \sum_{i=1}^{n} r_i$ and $b(u) = \sum_{i=1}^{n} s_i$. Note that if $u, v \in I(S)$ with $v \leqslant' u$ in G, then $a(v) \leqslant a(u)$ and $b(u) \leqslant b(v)$.

LEMMA 2.9. *Let $u = a^{r_1} b^{-s_1} \cdots a^{r_n} b^{-s_n}$ and $v = a^{p_1} b^{-q_1} \cdots a^{p_m} b^{-q_m}$ be reduced in $I(S)$, and suppose $v \leqslant' u$ in G. Let v_0 be the shortest initial segment of v for which $b(v_0) = b(u)$. Then in $E_I = E_{I(n)}$ we have $u * v_0 \leqslant a^{a(u)-a(v_0)}$.*

The idea is to apply Lemmas 2.7 and 2.8 alternately. At each application, either a power of a or a power of b is removed, respectively, and since $v \leqslant' u$ we achieve the result. A typical situation is illustrated



Proof. If $b(v_0) = 0$ the result follows from Lemma 2.7; if $a(v_0) = 0$ it follows from Lemma 2.8. So we may assume $a(v_0) \geqslant 1$ and $b(v_0) \geqslant 1$. There are two cases.

(i) $p_1 \geqslant 1$. We use induction on $a(v_0)$. If $a(v_0) = 1$ then $v_0 = ab^{-k}$, say, and by Lemma 2.7 there exists a reduced word u' with $u * v_0 = u * a * b^{-k} \leqslant u' * b^{-k}$, where $a(u') = a(u) - 1$, $b(u') = k$. By Lemma 2.8, we have $u' * b^{-k} \leqslant a^{a(u')} = a^{a(u)-a(v_0)}$.

Suppose the result holds for $a(v_0) = t > 1$. Then again by Lemma 2.7, there exists a reduced word u' with, writing $v_0 = av'$,

$$u * v_0 = u * a * v' \leqslant u' * v',$$

where $a(u') = a(u) - 1 \geqslant a(v_0) - 1 = a(v')$ and $b(u') = b(v')$.

By the induction hypothesis,

$$u * v_0 \leqslant u' * v' \leqslant a^{a(u')-a(v')} = a^{a(u)-a(v_0)},$$

whence the result.

(ii) $p_1 = 0$. Then $q_1 \geqslant 1$ and an argument similar to that in case (i) gives the result

COROLLARY 2.10. $u \wedge v \leqslant v_0 a^{a(u)-a(v_0)}$ *in* E_I.

Proof. Write $v = v_0 v_1$. Then

$$\begin{aligned} u \wedge v &= u * vv^{-1} = u * v_0 v_1 v_1^{-1} v_0^{-1} = u * v_0 * v_1 v_1^{-1} v_0^{-1} \\ &\leqslant a^{a(u)-a(v_0)} * v_1 v_1^{-1} v_0^{-1} \leqslant a^{a(u)-a(v_0)} * v_0^{-1} = v_0 a^{a(u)-a(v_0)}. \end{aligned}$$

LEMMA 2.11. *In $I(S)$, for any nonnegative integers m and n,*

$$a^m \wedge b^{-n} = a^m b^{-n} \wedge b^{-n} = b^{-n} a^m \wedge a^m.$$

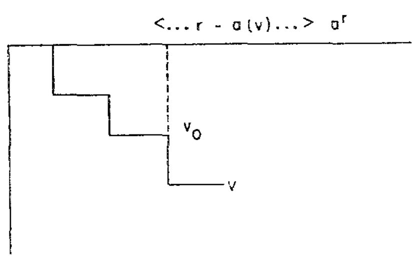
Proof. Since $a^{-m} b^{-n} = b^{-n} a^{-m}$,

$$a^m b^{-n} b^n a^{-m} b^{-n} b^n = a^m a^{-m} b^{-n} b^n;$$

similarly for $b^{-n} a^m \wedge a^m$.

LEMMA 2.12. *Let $v = a^{p_1} b^{-q_1} \cdots a^{p_m} b^{-q_m}$ be reduced in $I(S)$, with $a(v) \leqslant r$. Then $v \wedge a^r \leqslant v a^{r-a(v)}$ in E_I.*

Typically,



Proof:

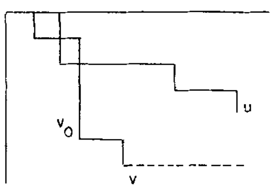
$$
\begin{aligned}
v \wedge a^r &= v * a^r * a^{-r} \\
&\leqslant (a^{-r} \wedge a^{-(r-a(v))}b^{-b(v)}) * a^{-r} \quad \text{(by Lemma 2.7)} \\
&= a^r \wedge a^{a(v)}b^{-b(v)} = (a^{r-a(v)} \wedge b^{-b(v)}) * a^{-a(v)} \\
&= (a^{r-a(v)}b^{-b(v)} \wedge b^{-b(v)}) * a^{-a(v)} \quad \text{(by Lemma 2.11)} \\
&\leqslant a^r b^{-b(v)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
v \wedge a^r = v \wedge v \wedge a^r \leqslant v \wedge a^r b^{-b(v)} &\\
\leqslant v_0 a^{r-a(v_0)} &\quad \text{(by Corollary 2.10)} \\
= v_0 a^{a(v)-a(v_0)} a^{r-a(v)} &= v a^{r-a(v)}.
\end{aligned}
$$

THEOREM 2.13. *Let $u = a^{r_1}b^{-s_1} \cdots a^{r_n}b^{-s_n}$ and $v = a^{k_1}b^{-q_1} \cdots a^{p_m}b^{-q_m}$ be reduced in $I(S)$, with $v \leqslant' u$ in G. Then $u \wedge v \leqslant va^{a(u)-a(v)}$.*

Typically,



Proof. With the notation already in use, we have

$$
\begin{aligned}
u \wedge v &\leqslant v \wedge v_0 a^{a(u)-a(v_0)} \quad \text{(by Corollary 2.10)} \\
&= v_0 v_1 \wedge v_0 a^{a(u)-a(v_0)} = (v_1 \wedge a^{a(u)-a(v_0)}) * v_0^{-1} \\
&\leqslant v_1 a^{a(u)-a(v_0)-a(v_1)} * v_0^{-1} \quad \text{(by Lemma 2.12)} \\
&= v_0 v_1 a^{a(u)-a(v)} = v a^{a(u)-a(v)}.
\end{aligned}
$$

By examining $u \wedge v = v * uu^{-1}$, one may use an exactly dual line of reasoning to prove that

$$u \wedge v \leqslant ub^{-b(v)+b(u)}.$$

Briefly, the procedure is to define u_0 to be the shortest initial segment of u for which $a(u_0) = a(v)$. Analogously to Lemma 2.9, it follows that $v * u_0 \leqslant b^{-b(v)+b(v_0)}$ and as in Corollary 2.10, that $u \wedge v \leqslant u_0 b^{-b(v)+b(v_0)}$. As in Lemma 2.12, one deduces that $u \wedge b^{-s} \leqslant ub^{-s+b(u)}$ when $b(u) \leqslant s$, and $u \wedge v \leqslant b^{-b(v)+b(u)}$ follows as in Theorem 2.13.

Combining these remarks with Theorem 2.13, we have the following theorem.

THEOREM 2.14. *Let* $u = a^{r_1}b^{-s_1} \cdots a^{r_n}b^{-s_n}$ *and* $v = a^{p_1}b^{-q_1} \cdots a^{p_m}b^{-q_m}$ *be reduced in* $I(S)$, *with* $v \leqslant' u$ *in* G. *Then*

$$u \wedge v \leqslant va^{a(u)-a(v)} \wedge ub^{-b(v)+b(u)}.$$

By interchanging a and b in Theorem 2.14 we have an analogous result for reduced words $u = b^{r_1}a^{-s_1} \cdots b^{r_n}a^{-s_n}$ and $v = b^{p_1}a^{-q_1} \cdots b^{-q_m}a^{-s_m}$ in $I(S)$, with $v \leqslant' u$ in G. Note that if $u = a^{r_1}b^{-s_1} \cdots a^{r_n}b^{-s_n}$ and $v = b^{p_1}a^{-q_1} \cdots b^{p_m}a^{-q_m} \in I(S)$ are comparable in G, then $u = v = 1$.

LEMMA 2.15. *Let* $v = a^{p_1}b^{-q_1} \cdots a^{p_m}b^{-q_m}$ *be reduced in* $I(S)$, *and suppose that* $\overline{\text{rlf.}\, v} \subseteq \bigvee \overline{\text{rlf.}\, u_j}$, *where* u_j *is reduced in* $I(S)$, $1 \leqslant j \leqslant n$. *Then* $\bigwedge u_j \leqslant v$ *in* E_I.

Proof. Since $\overline{\text{rlf.}\, w} = \overline{\text{rlf.}\, ww^{-1}}$ for any $w \in S \times S^{-1}$, we may consider v, u_j as elements of E_I; let $u = \bigwedge u_j$ in E_I.

We construct sets $U^0, U^1, U^2, \ldots, U^j, \ldots$ such that

$$\bigwedge_{U^j} u_i = u \leqslant \bigwedge v_i, \qquad 1 \leqslant i \leqslant j,$$

where $\{v_i\}$ are the initial segments $a^{p_1}, a^{p_1}b^{-q_1}, \ldots$ of v.

Define $U^0 = \{u_i : 1 \leqslant i \leqslant n\}$ and $U^{2j-1} = U^{2j-2} \cup \{a^{p_1}b^{-q_1} \cdots a^{p_j}\}$, $U^{2j} = U^{2j-1} \cup \{a^{p_1}b^{-q_1} \cdots a^{p_j}b^{-q_j}\}$ for $j = 1, 2, \ldots$. Since $\overline{\text{rlf.}\, v} \subseteq \bigvee \overline{\text{rlf.}\, u_j}$, there exist u_j, say u_1, and a left factor u_1' of u_1 such that $a^{p_1} \leqslant' u_1'$ in G. Then $u_1' = a^{p_1}x$, $x \in S$, so $u_1' = a^p b^{-q} \in S$ with $p_1 \leqslant p$. Hence $u_1' \leqslant a^{p_1}$ in E_1, and since u_1' is a left factor u_1, it follows that $a^{p_1} \geqslant u_1' \geqslant u_1 \geqslant \bigwedge u_i$, the intersection being over U^0. Hence

$$\bigwedge_{U^1} u_i = a^{p_1} \wedge \left(\bigwedge_{U^0} u_i\right) = \bigwedge_{U^0} u_i = u.$$

Next, $a^{p_1} \geqslant' a^{p_1}b^{-q_1} \in \overline{\text{rlf.}\, v}$, so there exist u_2, say, and a left factor u_2' of u_2, such that $u_2' \leqslant' a^{p_1}b^{-q_1} \leqslant' a^{p_1}$. By Theorem 2.14,

$$a^{p_1} \wedge u_2' \leqslant a^{p_1}b^{-b(u_2')} \leqslant a^{p_1}b^{-q_1}.$$

As before, $a^{p_1} \wedge u_2 \leqslant a^{p_1}b^{-q_1}$ and so

$$\bigwedge_{U^2} u_i = a^{p_1}b^{-q_1} \wedge \left(\bigwedge_{U^1} u_i\right) = a^{p_1} \wedge \left(\bigwedge_{U^0} u_i\right) = u.$$

Hence $u \leqslant a^{p_1} \wedge a^{p_1}b^{-q_1}$.

Now suppose U^{2j-1} satisfies $\bigwedge_{U^{2j-1}} u_i = u \leqslant \bigwedge v_i$, $1 \leqslant i \leqslant 2j-1$, for $j > 1$. Since $v_{2j-1} \geqslant' v_{2j} \in \overline{\text{rlf.}\, v}$ we have that $u'_{2j} \leqslant' v_{2j} \leqslant' v_{2j-1}$ for some left factor u'_{2j} of, say, u_{2j}. By Theorem 2.14

$$u_{2j} \wedge v_{2j-1} \leqslant u'_{2j} \wedge v_{2j-1} \leqslant v_{2j-1} b^{-b(u'_{2j})+b(v_{2j-1})}$$

$$= v_{2j-1} b^{-q_j} b^{-b(u'_{2j})+b(v_{2j-1})+q_j}$$

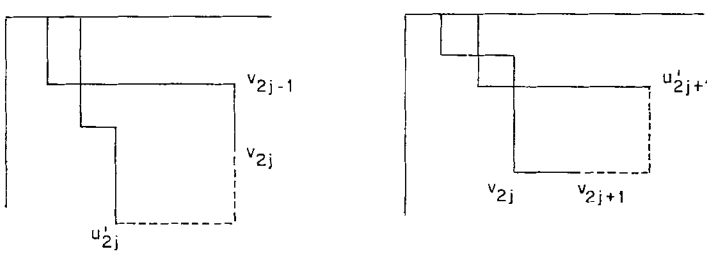
since

$$q_j \leqslant b(u'_{2j}) - b(v_{2j-1}) = v_{2j} b^{-b(u'_{2j})+b(v_{2j-1})+q_j} \leqslant v_{2j}.$$

Therefore $u_{2j} \wedge v_{2j-1} \wedge v_{2j} = u_{2j} \wedge v_{2j-1}$ and

$$\bigwedge_{U^{2j}} u_i = v_{2j} \wedge \left(\bigwedge_{U^{2j-1}} u_i \right) = \bigwedge_{U^{2j-1}} u_i = u \leqslant \bigwedge v_i, \qquad 1 \leqslant i \leqslant 2j. \quad (4)$$

Typically,



Similarly, $v_{2j} \leqslant' v_{2j+1} \leqslant' u'_{2j+1}$ for some left factor u'_{2j+1} of, say, u_{2j+1}. It follows that $u_{2j+1} \wedge v_{2j} \wedge v_{2j+1} = u_{2j+1} \wedge v_{2j}$, and that if U^{2j} satisfies (4) then

$$\bigwedge_{U^{2j+1}} u_i = v_{2j+1} \wedge \left(\bigwedge_{U^{2j}} u_i \right) = \bigwedge_{U^{2j}} u_i = u \leqslant \bigwedge v_i, \qquad 1 \leqslant i \leqslant 2j+1.$$

Thus we have constructed the sets U_j inductively and, on considering U_{2m}, have completed the proof.

THEOREM 2.16. *Let S be the free monoid on two commuting generators and let $\mathscr{Z}$ be the set of left translates of the closures of zig-zags in G, the free group on two commuting generators. Then*

$$I(S) \approx P(G, \mathscr{Z}, \{\overline{1}\}).$$

Proof. The homomorphism $\theta : S * S^{-1} \to P = P(G, \mathscr{X}, \{\overline{1}\})$ defined by $w\theta = (\overline{\text{rlf.}\, w}, w)$ maps $S * S^{-1}$ onto P (Theorem 2.3), and if ϕ is the canonical homomorphism of $S * S^{-1} \to I(S)$ then $\phi \circ \phi^{-1} \subseteq \theta \circ \theta^{-1}$. To prove that $P \approx I(S)$, we need only show that $\theta \circ \theta^{-1} \subseteq \phi \circ \phi^{-1}$.

Suppose that $u\theta = v\theta$ for $u, v \in S * S^{-1}$. Then $u = v$ in G, the maximum group homomorphic image of $I(S)$, and since $I(S)$ is proper (Proposition 2.5), to prove that $u\phi = v\phi$ it is enough to show $(u\phi)(u\phi)^{-1} = (v\phi)(v\phi)^{-1}$. For simplicity, we shall write $w\phi$ as w.

We prove in fact that $\overline{\text{rlf.}\, v} \subseteq \overline{\text{rlf.}\, u}$ implies $uu^{-1} \leqslant vv^{-1}$ in E_I. By Lemma 2.6 we may assume that

$$\overline{\text{rlf.}\, v} = \overline{\text{rlf.}\, vv^{-1}} \subseteq \bigvee_j \overline{\text{rlf.}\, u_j u_j^{-1}},$$

where u_j is reduced, $1 \leqslant j \leqslant m$, and again by Lemma 2.6, we may assume that v itself is reduced. As before, we shall consider v, u_j as elements of E_I.

Suppose then that $\overline{\text{rlf.}\, v} \subseteq \bigvee \overline{\text{rlf.}\, u_j}$ where v and u_j, $1 \leqslant j \leqslant m$, are reduced, and $u = \bigwedge u_j$ in E_I.

If $v = a^r b^s$ for $r, s \geqslant 0$, there exist u_i and a left factor u_i' of u_i such that $a^r b^s \leqslant u_i'$ in G. Then $u_i' = a^r b^s x$, $x \in S$, so $u_i' = a^p b^q$ with $r \leqslant p$, and it follows that $u = \bigwedge u_j \leqslant u_i \leqslant u_i' \leqslant v$ in E_I. Similarly, if $v = a^{-r} b^{-s}$ with r, $s \geqslant 0$, it follows that $u_i' = a^{-p} b^{-q}$ with $r \leqslant p$, and again $u \leqslant v$.

Finally, if $v = a^{p_1} b^{-q_1} \cdots a^{p_m} b^{-q_m}$, then $u \leqslant v$ in E_I by Lemma 2.15. Interchanging a and b, the same holds for $v = b^{p_1} a^{-q_1} \cdots b^{p_m} a^{-q_m}$.

Dually, $\overline{\text{rlf.}\, u} \subseteq \overline{\text{rlf.}\, v}$ implies $v \leqslant u$, and so $u = v$ in E_I. Thus $\theta \circ \theta^{-1} \subseteq \phi \circ \phi^{-1}$ and we conclude that $I(S) \approx P$.

In Theorem 6.2 of [4] it was shown that if $\mathscr{X}_u$ is the family of finitely generated up ideals of G, then $P(G, \mathscr{X}_u, [1, \to))$ is a simple inverse semigroup. It is the quotient of $S * S^{-1}$ modulo the congruence defined by $w_1 \equiv w_2$ if and only if $w_1 = w_2$ in G and w_1, w_2 generate the same up ideal in G. Dually, $P(G, \mathscr{X}_d, (\leftarrow, 1])$ is a simple inverse semigroup, the quotient of $S * S^{-1}$ modulo $w_1 \equiv w_2$ if and only if $w_1 = w_2$ in G and w_1, w_2 generate the same down ideal in G. Since $\overline{\text{rlf.}\, w}$ is the intersection of the up and down ideals generated by w, we have the following.

THEOREM 2.17. *$I(S)$ is a subdirect product of simple inverse semigroups.*

Recently, H. E. Scheiblich [7] determined the structure of the free monoid I_X on a set X. He showed that I_X is an F-inverse semigroup [5] and is, therefore, proper. However, although $I(S)$ is proper, it is not an F-inverse semigroup. For example, $(\overline{\text{rlf.}\, ab^{-1}}, ab^{-1})\sigma = (\overline{\text{rlf.}\, b^{-1}a}, b^{-1}a)\sigma$, but the σ-class of $(\overline{\text{rlf.}\, ab^{-1}}, ab^{-1})$ does not contain a maximum element.

3. Free Inverse Semigroups on More Than Two Commuting Generators

In this section, we show that the results of Section 2 do not extend to the free inverse monoid I_n on $n \geqslant 3$ commuting generators. Indeed the latter semigroups seem to be very different from I_1 and I_2.

The following notation will be used in this section. For a nonempty set X,

FI_X is the free inverse semigroup on X,

CI_X is the free inverse semigroup with commuting generators X,

F_X is the free semigroup on X.

From the universal properties of S^1 and CI_X, it is easy to see that, if $|X| = n$, $CI_X^1 = I_n$.

Consider the following elementary transformations on the elements of S $S = F_X * F_X^{-1}$;

$$\begin{array}{ll} ux_ix_jv \to ux_jx_iv & C\text{-steps} \\ \left.\begin{array}{l} uwv \to uww^{-1}wv \\ uww^{-1}wv \to uwv \\ uww^{-1}zz^{-1}v \to uzz^{-1}ww^{-1}v \end{array}\right\} & I\text{-steps} \end{array}$$

for $u, v \in S^1$, $w, z \in S$ and $x_i, x_j \in X$.

Lemma 3.1. *Let X be a nonempty set and let $w_1, w_2 \in F_X * F_X^{-1}$;*

(i) *$w_1 = w_2$ in $FI_X \Leftrightarrow w_1$ can be changed into w_2 by a finite sequence of I-steps;*

(ii) *$w_1 = w_2$ in $CI_X \Leftrightarrow w_1$ can be changed into w_2 by a finite sequence of C and I-steps.*

Proof. The assertion in (i) is Vagner's characterization of FI_X [9]. The assertion in (ii) follows from that in (i) since the only further relation imposed is the commutativity of the generators.

A word $w \in F_X * F_X^{-1}$ is *singular* if each syllable consists of a single letter. Thus, if w is singular, it cannot be altered by a C-step.

Lemma 3.2. *Let X be a nonempty set and let w_1 and $w_2 \in F_X * F_X^{-1}$, with w_1 singular. If $w_1 \to w_2$ by an I-step, then w_2 is singular.*

Proof. This follows straightforwardly from an examination of the various types of I-step.

Lemmas 3.1 and 3.2 combine to prove the following proposition which is fundamental in the remainder of this section.

PROPOSITION 3.3. *Let X be a nonempty set and let w_1, $w_2 \in F_X * F_X^{-1}$ with w_1 singular. Then $w_1 = w_2$ in CI_X if and only if $w_1 = w_2$ in FI_X; if this is the case, w_2 is singular.*

Proof. Suppose that $w_1 = u_0 \to u_1 \to \cdots \to u_n = w_2$ is a sequence of C-steps and I-steps. Then, since $u_0 = w_1$ is singular, the first step must be an I-step and then, by Lemma 3.2, u_1 is singular. Repeating the process, we find that each step is an I-step and w_2 is singular. Hence $w_1 = w_2$ in FI_X.

Conversely, since CI_X is quotient of FI_X, $w_1 = w_2$ in FI_X implies $w_1 = w_2$ in CI_X.

COROLLARY 3.4. *I_n, $n \geqslant 3$ is not proper.*

Proof. Let a, b, c be three of the free commuting generators for I_n. Then $w = ab^{-1}ca^{-1}bc^{-1}$ and w^2 are singular elements of $F_n * F_n^{-1}$. Since $w \neq w^2$ in the free group on n generators w and w^2 are not equal in FI_n. Hence, by Proposition 3.3, w is not an idempotent in CI_n and so, not in I_n.

The maximum group homomorphic image G of I_n is commutative so that $w = 1$ in G. Hence I_n is not proper.

Because I_n, $n \geqslant 3$ is not proper, it cannot be described by a construction analogous to that in Section 2. The next result shows that the idempotents in I_n, $n \geqslant 3$ are not the analogs of those in I_2.

COROLLARY 3.5. *Let X be a set with $n \geqslant 3$ elements and let w_1, $w_2 \in F_X * F_X^{-1}$. Then* solid hull rlf. w_1 = solid hull rlf. w_2 *need not imply $w_1w_1^{-1} = w_2w_2^{-1}$ in I_n.*

Proof. As in Corollary 3.4, let a, b, c be three of the free commuting generators for CI_X and set $w_1 = ab^{-1}ca^{-1}bc^{-1}$ and $w_2 = cb^{-1}ac^{-1}ba^{-1}$. Then

$$\text{rlf. } w_1 = \{1, a, ab^{-1}, ab^{-1}c, b^{-1}c, c\} = \text{rlf. } w_2$$

in the free commutative group on X. Hence solid hull rlf. w_1 = solid hull rlf. w_2.

On the other hand, $w_1w_1^{-1}$, $w_2w_2^{-1}$ are singular so they are equal in CI_X, and thus in I_n, if and only if they are equal in FI_X. The latter is not the case however for

$$\text{rlf. } w_1 = \{1, a, ab^{-1}, ab^{-1}c, ab^{-1}ca^{-1}, ab^{-1}ca^{-1}b, ab^{-1}ca^{-1}bc^{-1}\},$$

$$\text{rlf. } w_2 = \{1, c, cb^{-1}, cb^{-1}a, cb^{-1}ac^{-1}, cb^{-1}ac^{-1}b, cb^{-1}ac^{-1}ba^{-1}\}.$$

in the free group on X so that, by [8, Theorem 2.4] or [4, Theorem 5.5], $w_1w_1^{-1} \neq w_2w_2^{-1}$ in FI_X.

Given that it is not proper, it is natural to ask how far I_n, $n \geqslant 3$, is from being proper; equivalently, how large is the unitary subsemigroup

$$U_n = \{w \in I_n : (w, w^2) \in \sigma\}$$

of I_n ? We shall give an answer to this question by proving that, for $n \geqslant 3$, U_n contains the free inverse semigroup on two, and thus, by [7, Corollary 2.5], on countably many generators.

THEOREM 3.6. *Let* $X = \{a, b, c,...\}$ *and set* $w_1 = ab^{-1}ca^{-1}bc^{-1}$, $w_2 = bc^{-1}ab^{-1}ca^{-1}$ *in* CI_X. *Then the inverse subsemigroup* $\langle w_1, w_2\rangle$ *of* CI_X, *generated by* w_1 *and* w_2, *is free on* w_1 *and* w_2.

Proof. Let $W = \{w_1, w_2, w_1^{-1}, w_2^{-1}\}$, Then, since each element of W is singular and has even length, each element of $\langle w_1, w_2\rangle$ is singular and so, by Proposition 3.3, $\langle w_1, w_2\rangle$ is isomorphic to the subsemigroup of FI_X generated by w_1 and w_2. Hence without loss of generality, we can regard w_1, w_2 as being in FI_X.

To complete the proof of the theorem, one can verify that the hypotheses of [7], Proposition 2.3 are satisfied. However, the details, required to verify these hypotheses, can also be used to prove the result directly. We adopt the latter approach.

Let ϕ be the canonical homomorphism $FI_Y \to \langle w_1, w_2\rangle$, where $Y = \{w_1, w_2\}$. Then, in order to show that $\langle w_1, w_2\rangle$ is free, it suffices to prove that ϕ is idempotent separating; since [7, Lemma 1.3] $\mathscr{H}$ is trivial on free inverse semigroups, ϕ is then an isomorphism.

By [8, Theorem 2.4] or [4, Theorem 5.5], for $u, v \in \langle w_1, w_2\rangle$,

$$uu^{-1} = vv^{-1} \text{ in } FI_X \Leftrightarrow \text{rlf. } u = \text{rlf. } v$$

in the free group G_X on X while

$$uu^{-1} = vv^{-1} \text{ in } FI_Y \Leftrightarrow \text{rlf. } u = \text{rlf. } v$$

in G_Y, the free group on Y. Hence, to complete the proof of the theorem, it suffices to prove that rlf. u = rlf. v in G_X implies rlf. u = rlf. v in G_Y, for $u, v \in \langle w_1, w_2\rangle$. We do this by analyzing the reduction in G_X of words $g = g_1g_2 \cdots g_nh$ where $g_i \in W$, $1 \leqslant i \leqslant n$ and h is a proper initial segment of a member of W.

Consider the following elementary operations on words of the form

$$g = g_1g_2 \cdots g_nh \in F_X * F_X^{-1} \qquad \text{where} \quad g_i \in W, \quad 1 \leqslant i \leqslant n$$

and h is a proper initial segment of a member of W.

$g_1 \cdots g_i g_{i+1} \cdots g_n h \to g_1 \cdots g_{i-1} g_{i+2} \cdots g_n h$ if $g_{i+1} = g_i^{-1}$ B-step;

$g_1 \cdots g_n h \to g_1 \cdots g_{n-1} h'$ if h is an initial segment of g_n^{-1} and $g_n h = h'$ in G_X where h' is reduced E-step;

$g_1 \cdots g_i g_{i+1} \cdots g_n h \to g_1 \cdots g_{i-1} g_i' g_{i+2} \cdots g_n h$ if $g_i = w_2$, $g_{i+1} = w_1$ or $g_i = w_1^{-1}$, $g_{i+1} = w_2^{-1}$ and where $g_i' = g_i g_{i+1}$ is reduced in G_X;

$g_1 \cdots g_n h \to g_1 \cdots g_{n-1} g_n'$ if $g_n = w_2$ and h is an initial segment of w_1 or $g_n = w_1^{-1}$ and h is an initial segment of w_2^{-1} and where $g_n' = g_n h$ is reduced in G_X A-step.

We shall denote by gB the result of applying all possible B-steps to $g = g_1 g_2 \cdots g_n h$; gBE is the result of applying an E-step to gB; if such is possible, otherwise $gBE = gB$; $gBEA$ is the result of applying all possible A-steps to gBE.

LEMMA 3.7. *gBEA is reduced.*

Proof. $gB = g_1' \cdots g_s' h$ where $g_{i+1}' \neq (g_i')^{-1}$, $1 \leqslant i < s$. Thus $gBE = gB$ if h is not an initial segment, $\neq 1$, of $(g_s')^{-1}$ and $gBE = g_1' \cdots g_{s-1}' h'$ if h is an initial segment of $(g_s')^{-1}$ when $h' = g_s' h$ is an initial segment of g_s'. Because of the form of the elements of W, it follows that gBE can only be reduced further by removing subwords $a^{-1}a$ from subwords of the forms $w_2 w_1$, $w_1^{-1} w_2^{-1}$ or of the form $g_{s-1}' h'$ if $g_{s-1}' = w_2$ (w_1^{-1}) and h' is an initial segment of w_1 (w_2^{-1}) or of the form $g_s' h$ if $g_s' = w_2$ (w_1^{-1}) and h is an initial segment of w_1 (w_2^{-1}). That is, gBE can be reduced further only by applying A-steps. Hence $gBEA$ is reduced.

The idea of the proof of Theorem 3.6 is to construct rlf. u in the free group on w_1, w_2 from rlf. u in G_X. If $u = g_1 \cdots g_n$ where $g_i \in W$, $1 \leqslant i \leqslant n$ then

$$\text{rlf. } u \text{ in the free group on } w_1, w_2 = \{g_1 B, g_1 g_2 B, \ldots, g_1 \cdots g_n B\}.$$

Hence, to obtain this set from rlf. u in G_X, we must find some way of undoing the A-steps. This is accomplished by the following construction.

For any word $z \in F_X * F_X^{-1}$ define A^*-steps as follows:

$$z = pb^{-1}cb^{-1}q \to pb^{-1}ca^{-1}ab^{-1}q,$$
$$z = pc^{-1}bc^{-1}q \to pc^{-1}ba^{-1}abc^{-1}q,$$
$$z = pc^{-1}b \to pc^{-1}ba^{-1},$$
$$z = pb^{-1}c \to pb^{-1}ca^{-1},$$

and let zA^* be the result of applying all possible A^*-steps to z.

LEMMA 3.8. *If* $g = g_1 \cdots g_n h$ *then* $gBEAA^* = gBE$.

Proof. A^*-steps can be applied to $gBEA$ precisely when a subword $b^{-1}cb^{-1}$ or $c^{-1}bc^{-1}$ occurs in $gBEA$ or $c^{-1}b$ or $b^{-1}c$ occurs at the right end of $gBEA$. Let $gBE = g_1' \cdots g_s'h'$. Then from the form of w_1 and w_2 it is easy to see that we get terms $b^{-1}cb^{-1}$, $c^{-1}bc^{-1}$ in $gBEA$ or $c^{-1}b$, $b^{-1}c$ at the end of $gBEA$ precisely when an A-step is applied to gBE. Hence $gBEAA^* = gBE$.

COROLLARY 3.9. *Let* $u = g_1 \cdots g_n$ *where* $g_i \in W$, $1 \leqslant i \leqslant n$. If $g' \in$ rlf. u *in* G_X, *then* $g'A^*$ *has length* $6n$, *for some nonnegative integer* n, *if and only if* $g'A^* = (g_1 \cdots g_s)B$ *for some* $s \leqslant n$.

Proof. Suppose $g' = (g_1 \cdots g_s h)\, BEA$ where $g_1 \cdots g_s h$ is an initial segment of u. Then, by Lemma 3.8, $(g_1 \cdots g_s h)\, BE = g'A^*$. Now $(g_1 \cdots g_s h)\, BE$ has length a multiple of 6 if and only if h is absent. But, if h is absent, $(g_1 \cdots g_s h)\, BE = (g_1 \cdots g_s)B$.

Now to the proof of Theorem 3.6.

Let u, $v \in \langle w_1, w_2 \rangle$ and suppose rlf. $u =$ rlf. v in G_X. Then, with the obvious notation, (rlf. $u)A^* =$ (rlf. $v)A^*$ and so

$$(\text{rlf. } u)A^* \cap 6\mathbb{Z} = (\text{rlf. } v)A^* \cap 6\mathbb{Z}.$$

By Corollary 3.9,

(rlf. $u)A^* \cap 6\mathbb{Z} = \{g_1B, g_1g_2B, \ldots, g_1g_2 \cdots g_nB\} =$ rlf. u in the free group on w_1, w_2 where $u = g_1 \cdots g_n$ with $g_i \in W$, $1 \leqslant i \leqslant n$. Hence

rlf. u in the free group on w_1, $w_2 =$ rlf. v in the free group on w_1, w_2.

This shows that ϕ is idempotent separating and thus is an isomorphism.

The difference between I_1 and I_2 on the one hand and I_n, $n \geqslant 3$, on the other is pointed out again by Theorem 3.6 for we have the following proposition.

PROPOSITION 3.10. I_2 *does not contain the free inverse semigroup in two generators.*

Proof. Suppose that T is an inverse subsemigroup of I_2 and let $a, b \in T$. Then, from the multiplication in I_2, it is easy to see that $(a, b) \in \sigma$ if and only if $aa^{-1}bb^{-1}a = aa^{-1}bb^{-1}b$, where σ is the minimum group congruence on I_2; actually this is true for any proper inverse semigroup not just I_2. Hence $a = b$ in the maximum group homomorphic image of I_2 if and only if $a = b$ in the maximum group homomorphic image of T. Thus the maximum group homomorphic image of T can be embedded in the maximum group homomorphic image $\mathbb{Z} \times \mathbb{Z}$ of I_2. If T were free inverse on two generators this would not be possible since the maximum group homomorphic image of T would be the free group on two generators.

As a Corollary to Theorem 3.6, we have the following.

COROLLARY 3.11. *Every countably generated inverse semigroup is a homomorphic image of an inverse subsemigroup of* I_3.

REFERENCES

1. A. H. CLIFFORD AND G. B. PRESTON, "Algebraic Theory of Semigroups," Mathematical Surveys 7, Vol. 1 & 2, Providence, RI, 1961 & 1967.
2. D. B. MCALISTER, A homomorphism theorem for semigroups, *J. London Math. Soc.* **43** (1968), 355–366.
3. D. B. MCALISTER, Groups, semilattices and inverse semigroups, *Trans. Amer. Math. Soc.*, to appear.
4. D. B. MCALISTER AND R. MCFADDEN, Zig-zag representations and inverse semigroups, *J. Algebra* **32** (1974), 178–206.
5. R. MCFADDEN AND L. O'CARROLL, F-inverse semigroups, *Proc. London Math. Soc.* **22** (1971), 652–666.
6. W. D. MUNN, Uniform semilattices and bisimple inverse semigroups, *Quart. J. Math. (Oxford)* **17** (1966), 151–159.
7. N. R. REILLY, Free generators in free inverse semigroups, *Bull. Austral. Math. Soc.* **7** (1972), 407–424.
8. H. E. SCHEIBLICH, Free inverse semigroups, Announcement in Semigroup Forum 4 (1972), 351–359; *Proc. Amer. Math. Soc.* **38** (1973), 1–7.
9. V. V. VAGNER, Generalized heaps and generalized groups with a transitive compatibility relation, *Učen. Zap. Saratov Gos. Univ.* **70** (1961), 25–39.